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Abstract

In this paper we study the properties of any sequence $(u_n)_{n\geq 1}$ weakly converging to a nonnegative function u in $W_0^{1,p}(\Omega)$, $p>1$, and satisfying a variational inequality of type $-\operatorname{div}(a_n(\cdot,\nabla u_n)) \geq f_n$, where $(a_n)_{n\geq 1}$ is a suitable sequence of monotone operators and $(f_n)_{n\geq 1}$ is any strongly convergent sequence in the dual space $W^{-1,p'}(\Omega)$. We prove that the sequence $(u_n-(1-\varepsilon)\,u)^-$ strongly converges to 0 in $W_0^{1,p}(\Omega)$ for any $\varepsilon\in(0,1)$. We show by a counter-example that the result does not hold true if $\varepsilon=0$. A remarkable corollary of these strong ε-convergences is that the sequence $(u_n)_{n\geq 1}$ satisfies, up to a subsequence, a kind of semi-strong convergence: $(u_n)_{n\geq 1}$ can be bounded from below by a sequence which converges to the same limit u but strongly in $W_0^{1,p}(\Omega)$. We also give an example of a nonnegative weakly convergent sequence which does not satisfy this semi-strong convergence property and hence cannot satisfy any variational inequality of the previous type. Finally, in the linear case of a sequence of highly-oscillating matrices, we improve the strong ε-convergences by replacing the arbitrary small constant $\varepsilon>0$ by a sequence $(\varepsilon_n)_{n\geq 1}$ converging to 0.

1 Introduction

In [2] we proved the following result:

For any pair of sequences $(B_n,C_n)_{n\geq 1}$ of equi-coercive and bounded matrix-valued functions defined in a bounded open set Ω of $\mathbb{R}^d$, for any pair of sequences $(v_n,w_n)_{n\geq 1}$ weakly converging to (v,w) in $H_0^1(\Omega)^2$, and for any pair of strongly convergent sequences $(h_n,g_n)_{n\geq 1}$ in $H^{-1}(\Omega)^2$, such that for any $n\geq 1$,

$$-\operatorname{div}(B_n\nabla v_n)\geq g_n \ \text{ and } \ -\operatorname{div}(B_n\nabla v_n)\geq h_n \quad \text{in } \mathcal{D}'(\Omega), \tag{1.1}$$

we have the semi-continuity property

$$\forall\,\psi\in C^\infty(\bar\Omega),\ \psi\geq 0, \qquad \liminf_{n\to+\infty}\int_\Omega \psi\,B_n\nabla v_n\cdot\nabla w_n \geq \int_\Omega \psi\,B^*\nabla v\cdot\nabla w, \tag{1.2}$$

where B^* is the H-limit (well-defined up to a subsequence) of the sequence $(B_n)_{n\geq 1}$ in the sense of the H-convergence of Murat-Tartar [7].

One of the key-ingredient of the proof (1.2) is given by the following auxiliary result: Under the same assumptions for the sequences $(C_n)_{n\geq 1}$ and $(w_n)_{n\geq 1}$, if T_ε, for $\varepsilon > 0$, is a smooth ε-approximation of the function $(t \mapsto t^-)$ on $\mathbb{R}$ such that

$$T_\varepsilon(0) = 0 \qquad \text{and} \qquad \forall\, t \in \mathbb{R}, \quad \begin{cases} |T_\varepsilon(t) - t^-| \leq \varepsilon \\ -1 \leq T'_\varepsilon(t) \leq 0, \end{cases}$$

then we have the strong convergence

$$T_\varepsilon(w_n - w) \;\longrightarrow\; 0 \quad \text{in } H^1_0(\Omega). \tag{1.3}$$

Moreover, when $B_n = B$ is independent of n the strong convergence (1.3) holds with $T_\varepsilon(t) = t^-$. But in general, the sequence $(w_n - w)^-$ does not strongly converge to 0 because of the oscillations of B_n like in the homogenization theory (see Remark 3.6 of [2]). The proofs of (1.3) and (1.2) are rather technical and need a fine result of potential theory.

The purpose of this study is to give another and simpler strong convergence of type (1.3) under the extra assumption of nonnegativity of the limit and in a nonlinear framework. This new approach has a surprising consequence on the behaviour on the sequences satisfying a variational inequality such (1.1). The main result of the paper (see Theorem 2.1) is the following:

For any sequence $(a_n)_{n\geq 1}$ of uniformly p-monotone, $p > 1$, and uniformly bounded Carathéodory functions from $\Omega \times \mathbb{R}^d$ into $\mathbb{R}^d$ (see section 2 for details), for any sequence $(u_n)_{n\geq 1}$ weakly converging to $u \geq 0$ in $W^{1,p}_0(\Omega)$, and for any strongly convergent sequence $(f_n)_{n\geq 1}$ in $W^{-1,p'}(\Omega)$, $p' := \frac{p}{p-1} > 1$, such that for any $n \geq 1$,

$$-\operatorname{div}\left(a_n(\cdot, \nabla u_n)\right) \geq f_n \quad \text{in } \mathcal{D}'(\Omega), \tag{1.4}$$

we have the *semi-strong* convergences

$$\forall\, \varepsilon \in (0,1), \qquad (u_n - (1-\varepsilon)\, u)^- \;\longrightarrow\; 0 \quad \text{strongly in } W^{1,p}_0(\Omega). \tag{1.5}$$

Contrary to (1.3) we do not consider in (1.5) an ε-approximation of $(t \mapsto t^-)$. We keep this function but we have to introduce the shift εu inside to obtain the strong convergence. The price to pay is to assume the nonnegativity of the weak limit u.

A remarkable corollary of (1.5) (see Corollary 2.4) is that there exist a subsequence $(u_{\theta(n)})_{n\geq 1}$ and a sequence $(v_k)_{k\geq 1}$ strongly converging to u in $W^{1,p}_0(\Omega)$, such that

$$\forall\, n \geq k, \quad u_{\theta(n)} \geq v_k \quad \text{a.e. in } \Omega. \tag{1.6}$$

Thanks to inequality (1.6) the qualifying "semi-strong" for convergence (1.5) takes its whole meaning.

The strong convergences (1.5) are in some sense optimal since we cannot take $\varepsilon = 0$ if a_n depends actually on n. We give a counter-example (see Proposition 3.1) showing that the oscillations of a_n prevent from the strong convergence of $(u_n - u)^-$. The assumptions cannot be relaxed anymore. On the one hand, it is easy to check that the nonnegativity of u is a consequence of (1.5). On the other hand, the variational inequality (1.4) is a crucial assumption to obtain (1.5) and (1.6). Indeed, there exists a nonnegative sequence u_n weakly converging in $W^{1,p}_0(\Omega)$ which does not satisfy (1.6) and hence neither (1.5) (see Proposition 3.2). Such a sequence has the remarkable property to satisfy none variational inequality of type (1.4). This counter-example is quite general since it holds true provided that $p \leq d$.

If it is not possible to take $\varepsilon = 0$ in the strong convergence (1.5), a natural question is to know if we can replace the arbitrary small but fixed constant $\varepsilon > 0$ in (1.5) by a positive sequence $(\varepsilon_n)_{n\geq 1}$ converging to 0. We prove that the answer is positive (see Theorem 4.1) in the case of any sequence of highly-oscillating matrices, *i.e.* $a_n(x,\xi) = A(\frac{x}{\tau_n})\,\xi$, where A is a periodic matrix-valued function. In this linear framework we assume that the variational inequality (1.4) holds true and that u belongs to $W^{2,d}(\Omega)$. Then, there exists a positive sequence ε_n, with $\tau_n \ll \varepsilon_n \ll 1$, such that

$$(u_n - (1-\varepsilon_n)\,u)^- \longrightarrow 0 \quad \text{strongly in } W_0^{1,p}(\Omega). \tag{1.7}$$

The paper is organized as follows: The section 2 is devoted to the results (1.5) and (1.6) and to their proof. In section 3 we give two counter-examples showing the optimality of these results. In section 4 we prove the strong convergence (1.7) in the case of a sequence of highly-oscillating matrices.

2 Semi-strong convergence results

Let Ω be a bounded open set of $\mathbb{R}^d$, $d \geq 1$, let $p \in]1,+\infty[$ and $p' := \frac{p}{p-1}$. Let a be a fixed function from $\Omega \times \mathbb{R}^d$ into $\mathbb{R}^d$ which satisfies the following properties:

- a is a Carathéodory function, *i.e.*

$$\begin{cases} \text{a.e. } x \in \Omega, & a(x,\cdot) \text{ is continuous on } \mathbb{R}^d, \\[2mm] \forall\, \xi \in \mathbb{R}^d, & a(\cdot,\xi) \text{ is measurable on } \Omega; \end{cases}$$

- a is coercive, *i.e.* there exists a positive constant α and a nonnegative function γ in $L^1(\Omega)$ such that

$$\text{a.e. } x \in \Omega,\ \forall\, \xi \in \mathbb{R}^d, \quad a(x,\xi)\cdot\xi \geq \alpha\,|\xi|^p - \gamma(x);$$

- a is strictly monotone, *i.e.*

$$\text{a.e. } x \in \Omega,\ \forall\, \xi \neq \eta \in \mathbb{R}^d, \quad (a(x,\xi) - a(x,\eta))\cdot(\xi - \eta) > 0;$$

- a is bounded, *i.e.* there exists a positive constant β and a nonnegative function δ in $L^p(\Omega)$ such that

$$\text{a.e. } x \in \Omega,\ \forall\, \xi \in \mathbb{R}^d, \quad |a(x,\xi)| \leq \beta\,(|\xi| + \delta(x))^{p-1}\,.$$

Let $(a_n)_{n\geq 1}$ be a sequence of functions from $\Omega \times \mathbb{R}^d$ into $\mathbb{R}^d$ which satisfies the following properties:

(i) for any $n \geq 1$, a_n is a Carathéodory function,

(ii) a_n is uniformly monotone with respect to a, *i.e.*, for any $n \geq 1$,

$$\text{a.e. } x \in \Omega,\ \forall\, \xi,\eta \in \mathbb{R}^d, \quad (a_n(x,\xi) - a_n(x,\eta))\cdot(\xi-\eta) \geq (a(x,\xi) - a(x,\eta))\cdot(\xi-\eta);$$

(iii) a_n is uniformly bounded, *i.e.* there exists a positive constant β such that, for any $n \geq 1$,

$$\text{a.e. } x \in \Omega,\ \forall\, \xi \in \mathbb{R}^d, \quad |a_n(x,\xi)| \leq \beta\,|\xi|^{p-1}.$$

Under the previous assumptions we have the following result:

Theorem 2.1 *Let $(u_n)_{n\geq 1}$ be a sequence in $W_0^{1,p}(\Omega)$ such that*

$$u_n \rightharpoonup u \quad \textit{weakly in } W_0^{1,p}(\Omega). \tag{2.1}$$

Assume that there exists a strongly convergent sequence $(f_n)_{n\geq 1}$ in $W^{-1,p'}(\Omega)$ such that

$$-\operatorname{div}(a_n(\cdot, \nabla u_n)) \geq f_n \quad \textit{in } \mathcal{D}'(\Omega). \tag{2.2}$$

Then, we have the implication

$$u \geq 0 \textit{ a.e. in } \Omega \quad \Longrightarrow \quad \begin{cases} \forall\, v \in W_0^{1,p}(\Omega), \\ 0 \leq v \leq u \textit{ a.e. in } \Omega \quad \textit{and} \quad v < u \textit{ e. in } \{u > 0\}, \\ (u_n - v)^- \;\longrightarrow\; 0 \textit{ strongly in } W_0^{1,p}(\Omega). \end{cases} \tag{2.3}$$

(a.e. for almost everywhere and e. for everywhere).

Remark 2.2 It is easy to check that the function $v := (1-\varepsilon)\,u$ satisfies the requirements of (2.3) for any $\varepsilon \in (0,1)$. In this case, we obtain the equivalence

$$u \geq 0 \text{ a.e. in } \Omega \iff \forall\, \varepsilon \in (0,1),\;\; (u_n - (1-\varepsilon)\,u)^- \;\to\; 0 \text{ strongly in } W_0^{1,p}(\Omega). \tag{2.4}$$

The implication $(\Rightarrow)$ is an immediate consequence of (2.3) with $v := (1-\varepsilon)\,u$. Inversely, assume that the right hand side of (2.4) holds true. Then, the sequence $(u_n - (1-\varepsilon)\,u)^-$ weakly converges to εu^- and strongly to 0 in $W_0^{1,p}(\Omega)$. Therefore, the uniqueness of the weak limit in $W_0^{1,p}(\Omega)$ implies that $u^- = 0$ a.e. in Ω, or equivalently, $u \geq 0$ a.e. in Ω.

In the sequel, we will only focus on the strong convergences (2.4).

Remark 2.3 The variational inequality (2.2) implies the strong convergence of the negative part of the sequence $(u_n - u)$, up to an arbitrary small shift εu. In [2] we proved that the strong convergence holds with $\varepsilon = 0$, without assuming the nonnegativity of u but assuming that a_n does not depend on n. In general, the sequence $(u_n - u)^-$ does not strongly converge to zero in $W_0^{1,p}(\Omega)$, even if $u \geq 0$ a.e. in Ω. This is due to the oscillations effects of the sequence a_n (see Proposition 3.1 below). Moreover, inequality (2.2) cannot be relaxed (see Proposition 3.2 below).

The previous semi-strong convergence result allows us to obtain a strong approximation from below of the sequence u_n:

Corollary 2.4 *Let $(u_n)_{n\geq 1}$ be a sequence in $W_0^{1,p}(\Omega)$ which satisfies assumptions* (2.1) *with $u \geq 0$ a.e., and* (2.2)*. Then, there exist a subsequence $(u_{\theta(n)})_{n\geq 1}$ and a sequence $(v_k)_{k\geq 1}$ strongly converging to u in $W_0^{1,p}(\Omega)$, such that*

$$\forall\, n \geq k, \quad u_{\theta(n)} \geq v_k. \tag{2.5}$$

Proof of Theorem 2.1. Assume that $u \geq 0$ a.e. in Ω. Let v be a function in $W_0^{1,p}(\Omega)$ such that $0 \leq v \leq u$ a.e. in Ω and $v < u$ everywhere in $\{u > 0\}$. Set $E_n := \{u_n - v < 0\}$. By using successively the uniform monotonicity (ii) of a_n, the variational inequality (2.2) and the strong convergence of f_n to f in $W^{-1,p'}(\Omega)$, we have

$$
\begin{aligned}
0 &\le -\int_\Omega (a(x,\nabla u_n)-a(x,\nabla v))\cdot\nabla(u_n-v)^-\,dx\\
&=\int_\Omega (a(x,\nabla u_n)-a(x,\nabla v))\cdot\nabla(u_n-v)\,\mathbf{1}_{E_n}\,dx\\
&\le\int_\Omega (a_n(x,\nabla u_n)-a_n(x,\nabla v))\cdot\nabla(u_n-v)\,\mathbf{1}_{E_n}\,dx\\
&=-\int_\Omega (a_n(x,\nabla u_n)-a_n(x,\nabla v))\cdot\nabla(u_n-v)^-\,dx\\
&\le\int_\Omega a_n(x,\nabla v)\cdot\nabla(u_n-v)^-\,dx-\langle f_n,(u_n-v)^-\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}\\
&=\int_\Omega a_n(x,\nabla v)\cdot\nabla(u_n-v)^-\,dx-\langle f,(u-v)^-\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}+o(1)\\
&=\int_\Omega a_n(x,\nabla v)\cdot\nabla(u_n-v)^-\,dx+o(1)\quad(\text{since }(u-v)^-=0\text{ a.e. in }\Omega)\\
&=-\int_\Omega a_n(x,\nabla v)\cdot\nabla(u_n-v)\,\mathbf{1}_{E_n}\,dx+o(1).
\end{aligned}
\tag{2.6}
$$

Moreover, using the boundedness (iii) of a_n, the boundedness of u_n in $W_0^{1,p}(\Omega)$ and the Hölder inequality yields

$$
\left|\int_\Omega a_n(x,\nabla v)\cdot\nabla(u_n-v)\,\mathbf{1}_{E_n}\,dx\right|\le c\left(\int_\Omega|\nabla v|^p\,\mathbf{1}_{E_n}\,dx\right)^{\frac{1}{p'}}. \tag{2.7}
$$

Since $v\ge 0$ a.e. in Ω, we have $\nabla v\,\mathbf{1}_{E_n}=\nabla v\,\mathbf{1}_{E_n\cap\{v>0\}}$ a.e. in Ω. Let E be the subset of Ω composed of the x satisfying the pointwise convergence $u_n(x)\to u(x)$ and the inequality $v(x)\le u(x)$. Up to an extraction of a subsequence, still denoted by n, the set $\Omega\setminus E$ has a zero Lebesgue measure. Let $x\in E$. Assume by contradiction that there exists a subsequence n' such that $x\in E_{n'}\cap\{v>0\}$, for any $n'\ge 1$. Then, passing to the limit in the inequality $u_{n'}(x)<v(x)$ yields $u(x)\le v(x)$, and consequently, $u(x)=v(x)$. Since $v(x)>0$, we also have $u(x)>0$ and thus $v(x)<u(x)$ by the assumption on v, which establishes a contradiction. So, any $x\in E$ belongs to a finite number of sets $E_n\cap\{v>0\}$, $n\ge 1$. Therefore, the sequence $\mathbf{1}_{E_n\cap\{v>0\}}$ converges to 0 a.e. in Ω. The Lebesgue dominated convergence theorem thus implies that the right hand side of (2.7) tends to 0. This combined with estimate (2.6) implies that

$$
\int_\Omega (a(x,\nabla u_n)-a(x,\nabla v))\cdot\nabla(u_n-v)^-\,dx\underset{n\to+\infty}{\longrightarrow}0. \tag{2.8}
$$

Finally, following the first step of the proof of Theorem 2.19 in [2], we deduce from convergence (2.8) and the properties of a, the strong convergence of (2.3).

Proof of Corollary 2.4. By the strong convergences (2.4) of Remark 2.2, for each integer $k\ge 1$, the sequence $\nabla\left(u_n-(1-k^{-1})\,u\right)$ strongly converges to 0 in $L^p(\Omega)^d$. Therefore, there exists a subsequence $\theta_k(n)$ of n such that

$$
\forall n\ge 1,\quad\left\|\nabla\left(u_{\theta_k(n)}-(1-k^{-1})\,u\right)^-\right\|_{L^p(\Omega)}\le\frac{1}{2^n}.
$$

We may also assume that $\theta_{k+1}(n)$ is a subsequence of $\theta_k(n)$, for any $k\ge 1$. Then, by considering the diagonal extraction $\theta(n):=\theta_n(n)$, we obtain, for any $n\ge k$, the equality

$\theta_n(n) = \theta_k(n_k)$ for some $n_k \geq n$, whence the estimate

$$\begin{aligned}
&\left\| \nabla \left(u_{\theta(n)} - (1-k^{-1})\, u\right)^- \right\|_{L^p(\Omega)} \\
&= \left\| \nabla \left(u_{\theta_k(n_k)} - (1-k^{-1})\, u\right)^- \right\|_{L^p(\Omega)} \leq \frac{1}{2^{n_k}} \leq \frac{1}{2^n}.
\end{aligned} \tag{2.9}$$

In particular, thanks to the Poincaré inequality the series $\sum_{n\geq k} \left(u_{\theta(n)} - (1-k^{-1})\, u\right)^-$ converges in $W_0^{1,p}(\Omega)$, for any $k \geq 1$. We can thus define, for each $k \geq 1$, the function

$$v_k := (1-k^{-1})\, u - \sum_{n\geq k} \left(u_{\theta(n)} - (1-k^{-1})\, u\right)^- \in W_0^{1,p}(\Omega).$$

On the one hand, in virtue of (2.9) we have

$$\| \nabla v_k - \nabla u \|_{L^p(\Omega)} \leq \frac{1}{k}\, \|\nabla u\|_{L^p(\Omega)} + \sum_{n\geq k} \frac{1}{2^n} \underset{k\to+\infty}{\longrightarrow} 0,$$

which implies that the sequence v_k strongly converges to u in $W_0^{1,p}(\Omega)$. On the other hand, we have, for any $n \geq k$,

$$\begin{aligned}
u_{\theta(n)} &= (1-k^{-1})\, u + \left(u_{\theta(n)} - (1-k^{-1})\, u\right)^+ - \left(u_{\theta(n)} - (1-k^{-1})\, u\right)^- \\
&\geq (1-k^{-1})\, u - \left(u_{\theta(n)} - (1-k^{-1})\, u\right)^- \geq v_k,
\end{aligned}$$

which yields (2.5) and concludes the proof.

3 Counter-examples

The first counter-example shows that in general one cannot take $\varepsilon = 0$ in the semi-strong convergence (2.4) of Theorem 2.1:

Proposition 3.1 *There exist a sequence $(a_n)_{n\geq 1}$ and a nonnegative sequence $(u_n)_{n\geq 1}$ which satisfy the assumptions (2.1) and (2.2), such that $(u_n - u)^-$ does not strongly converge to 0 in $W_0^{1,p}(\Omega)$.*

The second counter-example provides a nonnegative and weakly convergent sequence in $W_0^{1,p}(\Omega)$, for which the result of Corollary 2.4 and thus the one of Theorem 2.1 does not hold true:

Proposition 3.2 *Assume that $p \leq d$. Then, there exists a nonnegative weakly convergent sequence in $W_0^{1,p}(\Omega)$, such that inequality (2.5) is satisfied by none of its subsequences.*

Remark 3.3 For $p > d$ the situation is completely different. Indeed, let Ω be a smooth bounded open subset of $\mathbb{R}^d$ and let $(u_n)_{n\geq 1}$ be a sequence which weakly converges to u in $W^{1,p}(\Omega)$. Then, by the Morrey embedding theorem there exists a subsequence $(u_{\theta(n)})_{n\geq 1}$ which converges uniformly to u in Ω, and thus satisfies

$$\delta_k := \sup_{n\geq k} \left\| u_{\theta(n)} - u \right\|_{L^\infty(\Omega)} \underset{k\to+\infty}{\longrightarrow} 0.$$

Therefore, the sequences $(u_{\theta(n)})_{n\geq 1}$ and $(v_k := u - \delta_k)_{k\geq 1}$ satisfy inequality (2.5) without any assumption of type (2.2).

Proof of Proposition 3.1. The dimension is $d := 1$ and $\Omega := (0, 1)$. For each integer $n \geq 1$, let ρ_n be the function defined in $(0, 1)$ by

$$\rho_n(x) := \begin{cases} \dfrac{1}{2} & \text{if } x \in \left[\dfrac{k}{n}, \dfrac{k}{n} + \dfrac{1}{2n}\right[\\[2ex] \dfrac{3}{2} & \text{if } x \in \left[\dfrac{k}{n} + \dfrac{1}{2n}, \dfrac{k+1}{n}\right[\end{cases} \qquad \text{for } k \in \{0, \ldots, n-1\},$$

and let u_n be the solution of

$$\begin{cases} -\left(\rho_n^{-1} u_n'\right)' = 1 & \text{in } (0, 1) \\[2ex] u_n(0) = u_n(1) = 0. \end{cases}$$

The sequence u_n clearly satisifies the assumptions (2.1) and (2.2) of Theorem 2.1 in the linear case. The weak limit of u_n in $H_0^1((0, 1))$ is $u(x) := \frac{1}{2} x (1 - x)$.

An easy but rather long computation yields for any $x \in \left[\frac{p}{n}, \frac{p}{n} + \frac{1}{2n}\right]$, $p \in \{0, \ldots, n-1\}$,

$$u_n(x) - u(x) = -\frac{p}{8n^2} + \frac{1}{4}\left(x - \frac{p}{n}\right)\left(x + \frac{p}{n} - 1\right) + \frac{1}{8n}\int_0^x \rho_n(t)\, dt.$$

Therefore, if $x < \frac{1}{2}$ and $x \in \left[\frac{p}{n} + \frac{1}{4n}, \frac{p}{n} + \frac{1}{2n}\right]$, then $x + \frac{p}{n} - 1 < 2x - 1 < 0$, whence

$$u_n(x) - u(x) \leq \frac{1}{16n}(2x - 1) + \frac{3}{16n}\, x = \frac{1}{16n}(5x - 1).$$

In particular, we have

$$\{u_n - u < 0\} \supset \left(0, \frac{1}{5}\right) \cap \left(\bigcup_{k=0}^{n-1}\left[\frac{k}{n} + \frac{1}{4n}, \frac{k}{n} + \frac{1}{2n}\right]\right),$$

which implies

$$\liminf_{n \to +\infty} \left| \{u_n - u < 0\} \cap \left(0, \frac{1}{5}\right) \right| \geq \frac{1}{20}. \tag{3.1}$$

On the other hand, we have

$$\left(u_n'(x) - u'(x)\right)^2 = \left[(\rho_n(x) - 1)\left(\frac{1}{2} - x\right) + \frac{1}{8n}\rho_n(x)\right]^2 \geq \frac{1}{4}\left(\frac{1}{2} - x\right)^2 + O\left(\frac{1}{n}\right),$$

which combined with estimate (3.1) yields

$$\liminf_{n \to +\infty} \int_0^{\frac{1}{4}} \left(u_n' - u'\right)^2 \mathbf{1}_{\{u_n - u < 0\}}\, dx \geq \frac{1}{4^3} \times \frac{1}{20} > 0.$$

Therefore, $(u_n - u)^-$ does not strongly converge to 0 in $H_0^1((0, 1))$.

Proof of Proposition 3.2. Let $Y := (-\frac{1}{2}, \frac{1}{2})^d$. We denote by B_r the ball centered at the origin of radius $r > 0$. Let $R \in]0, \frac{1}{2}[$ and let $(R_n)_{n \geq 1}$ be a sequence in $]0, R[$ converging to 0. Let $\hat{V}_n$, for $n \geq 1$, be the unique solution in $W_\#^{1,p}(Y)$ (the set of the Y-periodic functions in $W_{\text{loc}}^{1,p}(\mathbb{R}^d)$) of

$$\begin{cases} \operatorname{div}\left(|\nabla \hat{V}_n|^{p-2} \nabla \hat{V}_n\right) = 0 & \text{in } B_R \setminus \bar{B}_{R_n} \\[2ex] \hat{V}_\varepsilon = 1 & \text{in } Y \setminus \bar{B}_R \\[2ex] \hat{V}_\varepsilon = 0 & \text{in } B_{R_n}. \end{cases} \tag{3.2}$$

Let $(\varepsilon_n)_{n\geq 1}$ be a positive sequence converging to 0. We consider the ε_n-rescaled function defined by

$$\hat{v}_n(x) := \hat{V}_n\left(\frac{x}{\varepsilon_n}\right), \quad \text{for } x \in \Omega. \tag{3.3}$$

The function $\hat{v}_n(x)$ was introduced in [3] (for $p = 2$) to obtain a capacitary effect in homogenization. The sequence $(\hat{v}_n)_{n\geq 1}$ satisfies the following result:

Lemma 3.4 *Assume that $p \leq d$ and set*

$$R_n := \begin{cases} \varepsilon_n^{\frac{p}{d-p}} & \text{if } p < d \\ \\ \exp\left(-\varepsilon_n^{\frac{p}{1-p}}\right) & \text{if } p = d. \end{cases} \tag{3.4}$$

Then, we have

$$\hat{v}_n \rightharpoonup 1 \quad \text{weakly in } W^{1,p}(\Omega). \tag{3.5}$$

Set $\omega_n := \{\hat{v}_n = 0\} \cap \Omega$. Then, there exists a positive constant C such that the following estimate holds

$$\forall\, v \in W_0^{1,p}(\Omega), \quad \left| \frac{1}{|B_{R_n}|} \int_{\omega_n} v - \int_\Omega v \right| \leq C\, \|\nabla v\|_{L^p(\Omega)}. \tag{3.6}$$

Let us prove that the result of Proposition 3.2 is satisfied under the assumptions of Lemma 3.4. Let φ be a nonnegative and non-zero function in $C_c^\infty(\Omega)$ and let consider $u_n := \varphi\, \hat{v}_n$ for $n \geq 1$. The sequence u_n is nonnegative and by (3.5) weakly converges to φ in $W_0^{1,p}(\Omega)$. Assume by contradiction that there exists a subsequence, still denoted by u_n, and a sequence v_k which strongly converges to φ in $W_0^{1,p}(\Omega)$, such that inequality (2.5) holds. Thanks to estimate (3.6) we have, for any $n \geq k$,

$$\left| \frac{1}{|B_{R_n}|} \int_{\omega_n} v_k - \int_\Omega v_k \right| \leq \left| \frac{1}{|B_{R_n}|} \int_{\omega_n} \varphi - \int_\Omega \varphi \right| + C\, \|\nabla v_k - \nabla\varphi\|_{L^p(\Omega)}.$$

Moreover, the regularity of φ and the asymptotic $|\omega_n| \sim |\Omega|\, |B_{R_n}|$ imply that

$$\lim_{n\to+\infty} \frac{1}{|B_{R_n}|} \int_{\omega_n} \varphi = \int_\Omega \varphi,$$

which combined with the strong convergence of v_k to φ in $W_0^{1,p}(\Omega)$ yields

$$\left| \frac{1}{|B_{R_n}|} \int_{\omega_n} v_k - \int_\Omega v_k \right| \leq o_n(1) + o_k(1), \tag{3.7}$$

where $o_n(1)$ (respectively $o_k(1)$) denotes a sequence converging to 0 as $n \to +\infty$ (respectively $k \to +\infty$). Then, by using inequality (2.5) and the fact that $u_n = 0$ in ω_n, we deduce from (3.7) that

$$\int_\Omega v_k \leq \frac{1}{|B_{R_n}|} \int_{\omega_n} u_n + o_n(1) + o_k(1) = o_n(1) + o_k(1). \tag{3.8}$$

Therefore, passing successively to the limits $n \to +\infty$ and $k \to +\infty$ in (3.8) implies that

$$\int_\Omega \varphi \leq 0,$$

which yields the contradiction.

Proof of Lemma 3.4.

Proof of (3.5)*:* The function $\hat{V}_n$ defined by (3.2) is radial in the set $B_R \setminus \bar{B}_{R_n}$. More precisely, we have, for any $r \in (R_n, R)$,

$$\hat{V}_n(r) := \begin{cases} 1 + \left(\dfrac{r^{\frac{p-d}{p-1}} - R^{\frac{p-d}{p-1}}}{R_n^{\frac{p-d}{p-1}} - R^{\frac{p-d}{p-1}}} \right) & \text{if } p < d \\ \\ 1 + \left(\dfrac{\ln r - \ln R}{\ln R - \ln R_n} \right) & \text{if } p = d, \end{cases}$$

whence there exists a positive constant $c_{d,p}$ independent of n such that

$$\|\nabla \hat{V}_n(r)\|^p_{L^p(Y)} = \begin{cases} c_{d,p} \left(R_n^{\frac{p-d}{p-1}} - R^{\frac{p-d}{p-1}} \right)^{1-p} & \sim \quad c_{d,p}\, R_n^{d-p} & \text{if } p < d \\ c_{d,p} \, (\ln R - \ln R_n)^{1-p} & \sim \quad c_{d,p}\, |\ln R_n|^{1-p} & \text{if } p = d. \end{cases}$$

This estimate combined with the choice (3.4) of R_n implies that the sequence $\hat{v}_n$ defined by (3.3) is bounded in $W^{1,p}(\Omega)$. Moreover, since $\hat{V}_n = 1$ in the set $Y \setminus \bar{B}_R$ the weak limit of $\hat{v}_n$ is 1, which yields (3.5).

Proof of (3.6)*:* Denote by S_r the sphere centered at the origin and of radius $r > 0$. Let $V \in C^1(\bar{Y})$ and let $\tilde{V}$ be the function defined in spherical coordinates by $\tilde{V}(r, \xi) := V(y)$, where $y = r\,\xi$ with $r > 0$ and $\xi \in S_1$. By starting from the equality

$$\tilde{V}(R, \xi) - \tilde{V}(R_n, \xi) = \int_{R_n}^{R} \frac{\partial \tilde{V}}{\partial r}(r, \xi)\, dr$$

and by using the Hölder inequality, we obtain the inequality

$$\left| \tilde{V}(R, \xi) - \tilde{V}(R_n, \xi) \right| \leq \alpha_n \left(\int_{R_n}^{R} \left| \frac{\partial \tilde{V}}{\partial r}(R, \xi) \right|^p r^{d-1}\, dr \right)^{\frac{1}{p}},$$

$$\text{where} \quad \alpha_n := \begin{cases} \left[\dfrac{p-1}{p-d} \left(R^{\frac{p-d}{p-1}} - R_n^{\frac{p-d}{p-1}} \right) \right]^{\frac{1}{p'}} & \text{if } p < d \\ [\ln R - \ln R_n]^{\frac{1}{p'}} & \text{if } p = d. \end{cases} \tag{3.9}$$

Then, integrating the previous inequality with respect to $\xi \in S_1$ and using the Hölder inequality with respect to the integral in ξ, imply

$$\left| \fint_{S_{R_n}} V - \fint_{S_R} V \right| \leq c\, \alpha_n\, \|\nabla V\|_{L^p(Y)}, \tag{3.10}$$

where $\fint$ denotes the average-value and c is a positive constant. On the other hand, using a scaling of order R_n in the Poincaré-Wirtinger type inequality

$$\left| \fint_{B_R} W - \fint_{S_R} W \right| \leq c\, \|\nabla W\|_{L^p(Y)}, \quad \text{with } W(y) := V(R_n y),$$

implies that

$$\left| \fint_{B_{R_n}} V - \fint_{S_{R_n}} V \right| \leq c\, R_n^{\frac{p-d}{p}}\, \|\nabla V\|_{L^p(R_n Y)} \leq c\, R_n^{\frac{p-d}{p}}\, \|\nabla V\|_{L^p(Y)}. \tag{3.11}$$

The following Poincaré-Wirtinger type inequality also holds true

$$\left| \fint_Y V - \fint_{S_R} V \right| \leq c \, \|\nabla V\|_{L^p(Y)}. \tag{3.12}$$

Then, combining estimate (3.10) with (3.11) and (3.12) yields

$$\left| \fint_{B_{R_n}} V - \fint_Y V \right| \leq c \left(\alpha_n + R_n^{\frac{p-d}{p}} + 1 \right) \|\nabla V\|_{L^p(Y)}, \tag{3.13}$$

where c is a positive constant independent of the function V. Let v be a function in $W_0^{1,p}(\Omega)$, extended by 0 in $\mathbb{R}^d \setminus \Omega$. Then, putting the function $V(y) := v(\kappa + \varepsilon_n y)$, for $\kappa \in \mathbb{Z}^d$, in estimate (3.13) and summing over $\kappa \in \mathbb{Z}^d$, give

$$\left| \frac{1}{|B_{R_n}|} \int_{\omega_n} v - \int_\Omega v \right| \leq c \left(\varepsilon_n \alpha_n + \varepsilon_n R_n^{\frac{p-d}{p}} + \varepsilon_n \right) \|\nabla v\|_{L^p(\Omega)}. \tag{3.14}$$

Moreover, by the definition (3.9) of α_n and the choice (3.4) of R_n the sequences $\varepsilon_n \alpha_n$ and $\varepsilon_n R_n^{\frac{p-d}{p}}$ are bounded. Therefore, (3.14) yields the desired estimate (3.6).

4 The case of highly-oscillating linear operators

We restrict ourselves to a sequence of linear operators defined by highly-oscillating matrix-valued functions in a bounded open set Ω of $\mathbb{R}^d$, $d \geq 1$.

Let $Y := (0,1)^d$, let A be a Y-periodic matrix-valued function on $\mathbb{R}^d$ and let α, β be two positive constants such that

$$\text{a.e. } y \in \mathbb{R}^d, \ \forall \xi \in \mathbb{R}^d, \quad A(y)\xi \cdot \xi \geq \alpha \, |\xi|^2 \ \text{ and } \ A(y)^{-1}\xi \cdot \xi \geq \beta^{-1} \, |\xi|^2.$$

Let $(\tau_n)_{n \geq 1}$ be a positive sequence converging to 0 and let $(A_n)_{n \geq 1}$ be the sequence of oscillating matrices defined by

$$A_n(x) := A\left(\frac{x}{\tau_n}\right) \quad \text{a.e. } x \in \Omega. \tag{4.1}$$

Let $(e_1, \ldots, e_d)$ be the canonic basis of $\mathbb{R}^d$. By [1] we know that A_n H-converges, in the sense of Murat-Tartar [7], to the constant matrix A^* defined by

$$A^* e_i := \int_Y A(y) \left(e_i - \nabla \chi_i(y) \right) dy, \quad \text{for } i \in \{1, \ldots, d\}, \tag{4.2}$$

where χ_i is the unique function in $H^1_\#(Y)$, with zero average-value in Y, solution of

$$\operatorname{div}\left(A \, e_i - A \nabla \chi_i \right) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^d). \tag{4.3}$$

Moreover, for any sequence u_n converging to u weakly in $H^1_0(\Omega)$ such that $\operatorname{div}(A_n \nabla u_n)$ is compact in $H^{-1}(\Omega)$, we define the so-called corrector

$$\bar{u}_n := u - \tau_n \sum_{i=1}^d \chi_i \left(\frac{x}{\tau_n} \right) \frac{\partial u}{\partial x_i}. \tag{4.4}$$

Indeed, if u is smooth enough the sequence $\bar{u}_n$ strongly converges to u in $H^1_{\text{loc}}(\Omega)$.

In this framework, Theorem 2.1 can be improved by the following way:

Theorem 4.1 *Let Ω be a regular (with Lipschitz boundary) bounded open set of $\mathbb{R}^d$, $d \geq 1$. Let $(u_n)_{n\geq 1}$ be a sequence weakly converging to u in $H^1_0(\Omega)$, such that*

$$u \geq 0 \text{ a.e. in } \Omega \quad \text{and} \quad u \in W^{2,d\vee 2}(\Omega), \tag{4.5}$$

where $d \vee 2$ denotes the maximum between d and 2. Assume that there exists a sequence $(f_n)_{n\geq 1}$ strongly converging in $H^{-1}(\Omega)$, such that

$$-\operatorname{div}(A_n \nabla u_n) \geq f_n \quad \text{in } \mathcal{D}'(\Omega). \tag{4.6}$$

Then, there exists a positive sequence $(\varepsilon_n)_{n\geq 1}$ converging to 0 such that

$$(u_n - (1-\varepsilon_n)\, u)^- \longrightarrow 0 \quad \text{strongly in } H^1_0(\Omega). \tag{4.7}$$

Proof of Theorem 4.1.
First, we need to modify the corrector (4.4) by introducing truncatures and a cut-off function:

$$\bar{u}_n := u - \tau_n \sum_{i=1}^{d} \psi_n(x)\, T_{k_n}(\chi_i)\left(\frac{x}{\tau_n}\right) T_{k_n}\left(\frac{\partial u}{\partial x_i}\right), \tag{4.8}$$

where T_k, for $k \in \mathbb{N}$, is the function defined by $T_k(t) := \max(-k, \min(k,t))$, for $t \in \mathbb{R}$, $(k_n)_{n\geq 1}$ is a sequence of positive integers which tends to $+\infty$, and $(\psi_n)_{n\geq 1}$ is a sequence of functions in $C^1_0(\Omega)$ satisfying, for any $n \geq 1$,

$$\begin{cases} 0 \leq \psi_n \leq 1 & \text{in } \Omega \\ \psi_n(x) = 1 & \text{if } \operatorname{dist}(x, \partial\Omega) > \eta_n, \quad \text{where } \eta_n \to 0, \\ |\nabla \psi_n| \leq c\, \eta_n^{-1} & \text{in } \Omega. \end{cases}$$

Such a sequence ψ_n exists since Ω is regular. So, the function $\bar{u}_n$ belongs to $H^1_0(\Omega)$.

The proof is then divided in two steps:

First step: $(u_n - \bar{u}_n)^-$ strongly converges to 0 in $H^1_0(\Omega)$.
We get rid of the cut-off function ψ_n by introducing the new function

$$\tilde{u}_n := u - \tau_n \sum_{i=1}^{d} T_{k_n}(\chi_i)\left(\frac{x}{\tau_n}\right) T_{k_n}\left(\frac{\partial u}{\partial x_i}\right). \tag{4.9}$$

We have

$$\begin{aligned} \nabla \tilde{u}_n - \nabla \bar{u}_n \;&= \tau_n \sum_{i=1}^{d} \nabla \psi_n(x)\, T_{k_n}(\chi_i)\left(\frac{x}{\tau_n}\right) T_{k_n}\left(\frac{\partial u}{\partial x_i}\right) \\ &+ \sum_{i=1}^{d} (\psi_n(x) - 1)\, \nabla T_{k_n}(\chi_i)\left(\frac{x}{\tau_n}\right) T_{k_n}\left(\frac{\partial u}{\partial x_i}\right) \\ &+ \tau_n \sum_{i=1}^{d} (\psi_n(x) - 1)\, T_{k_n}(\chi_i)\left(\frac{x}{\tau_n}\right) \nabla\left[T_{k_n}\left(\frac{\partial u}{\partial x_i}\right)\right]. \end{aligned} \tag{4.10}$$

Since $\chi_i \in W^{1,p}_{\#}(Y)$, for some $p > 2$, by the Meyers theorem [5], and since $\nabla u \in L^{\frac{2p}{p-2}}(\Omega)^d$ by (4.5) and the Sobolev embedding theorem, the first term of the right hand side of (4.10) is an $O(\tau_n\, \eta_n^{-1})$ in $L^2(\Omega)$-norm by the Hölder inequality. Similarly, the second term is an $O(\eta_n^{\gamma})$ in $L^2(\Omega)$-norm, for any $\gamma < \frac{p-2}{2p}$, by the Hölder inequality. Finally, since

$\nabla^2 u \in L^{d\vee 2}(\Omega)^{d\times d}$ by (4.5), the last term of (4.10) is an $O(\tau_n\, k_n)$ in $L^2(\Omega)$-norm. Then, choosing k_n and τ_n such that

$$\lim_{n\to+\infty} \tau_n\left(k_n+\eta_n^{-1}\right)=0,$$

yields

$$\nabla\tilde{u}_n-\nabla\bar{u}_n \longrightarrow 0 \quad \text{strongly in } L^2(\Omega)^d. \tag{4.11}$$

We are thus led to study the sequence $\nabla\tilde{u}_n$ which satisfies

$$\begin{aligned}\nabla\tilde{u}_n-\nabla u+\sum_{i=1}^d \nabla\chi_i\left(\frac{x}{\tau_n}\right)\frac{\partial u}{\partial x_i} &= \sum_{i=1}^d \nabla\left(\chi_i-T_{k_n}(\chi_i)\right)\left(\frac{x}{\tau_n}\right)\frac{\partial u}{\partial x_i}\\ &+\sum_{i=1}^d \nabla T_{k_n}(\chi_i)\left(\frac{x}{\tau_n}\right)\left[\frac{\partial u}{\partial x_i}-T_{k_n}\left(\frac{\partial u}{\partial x_i}\right)\right]\\ &-\tau_n\sum_{i=1}^d T_{k_n}(\chi_i)\left(\frac{x}{\tau_n}\right)\nabla\left[T_{k_n}\left(\frac{\partial u}{\partial x_i}\right)\right].\end{aligned} \tag{4.12}$$

Since $\nabla\chi_i\in L^p_\#(Y)^d$, for some $p>2$, and since $\nabla u\in L^{\frac{2p}{p-2}}(\Omega)$ by (4.5), the first term of the right hand side of (4.12) is bounded in $L^2(\Omega)$-norm by a constant times

$$\left\|\nabla\chi_i\,\mathbf{1}_{\{|\chi_i|>k_n\}}\right\|_{L^p(Y)},$$

which converges to 0 by the Lebesgue dominated convergence theorem. Similarly, the second term is bounded in $L^2(\Omega)$-norm by a constant times

$$\left\|\frac{\partial u}{\partial x_i}\,\mathbf{1}_{\left\{\left|\frac{\partial u}{\partial x_i}\right|>k_n\right\}}\right\|_{L^{\frac{2p}{p-2}}(Y)},$$

which also converges to 0. Finally, since $\nabla^2 u\in L^{d\vee 2}(\Omega)^{d\times d}$ by (4.5), the last term of (4.12) is an $O\left(\tau_n k_n\right)$ in $L^2(\Omega)^d$-norm. Therefore, by choosing k_n such that

$$\lim_{n\to+\infty}\tau_n\, k_n^2=0$$

(the square will be necessary below), estimate (4.11) and equality (4.12) imply the convergence

$$\nabla\bar{u}_n-\nabla u+\sum_{i=1}^d\nabla\chi_i\left(\frac{x}{\tau_n}\right)\frac{\partial u}{\partial x_i} \longrightarrow 0 \quad \text{strongly in } L^2(\Omega)^d. \tag{4.13}$$

Note that the convergence (4.13) combined with the Hölder type inequality

$$\left\|\sum_{i=1}^d\nabla\chi_i\left(\frac{x}{\tau_n}\right)\frac{\partial u}{\partial x_i}\right\|_{L^2(\Omega)}\le c\sum_{i=1}^d\|\nabla\chi_i\|_{L^p(Y)}\,\|\nabla u\|_{L^{\frac{2p}{p-2}}(Y)},$$

and the inequality $|\bar{u}_n-u|\le d\,\tau_n\,k_n^2$, imply that

$$\bar{u}_n\rightharpoonup u \quad \text{weakly in } H^1_0(\Omega). \tag{4.14}$$

On the other hand, following for example [4] (pages 26-27), by (4.3) and (4.2) there exists, for each $i\in\{1,\dots,d\}$, an antisymmetric matrix-valued function Φ_i in $H^1_\#(Y)^{d\times d}$ such that

$$(A\,e_i-A\nabla\chi_i)-A^*e_i=\operatorname{div}\Phi_i \quad \text{in } \mathcal{D}'(\mathbb{R}^d).$$

Then, by the definitions (4.1) of A_n, (4.8) of $\bar{u}_n$ and by the strong convergence (4.13) we have

$$\begin{aligned} A_n\nabla\bar{u}_n - A^*\nabla u \ &= \tau_n \sum_{i=1}^{d} \operatorname{div}\left[\frac{\partial u}{\partial x_i}\,\Phi_i\left(\frac{x}{\tau_n}\right)\right] \\ &\quad - \tau_n \sum_{i=1}^{d} \Phi_i\left(\frac{x}{\tau_n}\right)\nabla\left(\frac{\partial u}{\partial x_i}\right) + o(1), \end{aligned} \tag{4.15}$$

where $o(1)$ denotes a strongly convergent sequence to 0 in $L^2(\Omega)^2$. Since Φ_i is antisymmetric, the first term of the right hand side of (4.16) is divergence-free. Moreover, since $\nabla^2 u \in L^{d\vee 2}(\Omega)^{d\times d}$ and $\Phi_i \in L_{\#}^{\frac{2d}{d-2}}(Y)^{d\times d}$ by the Sobolev embedding theorem, the second term is an $O\left(\tau_n k_n\right)$ in $L^2(\Omega)$-norm, whence

$$\operatorname{div}\left(A_n\nabla\bar{u}_n\right) \longrightarrow \operatorname{div}\left(A^*\nabla u\right) \quad \text{strongly in } H^{-1}(\Omega). \tag{4.16}$$

Now, let us conclude the first step. Using successively the assumption (4.6), the weak convergence (4.14) and the strong one (4.16), yields

$$\begin{aligned} &\int_\Omega A_n\nabla(u_n-\bar{u}_n)^-\cdot\nabla(u_n-\bar{u}_n)^-\,dx \\ &= -\int_\Omega A_n\nabla(u_n-\bar{u}_n)\cdot\nabla(u_n-\bar{u}_n)^-\,dx \\ &\le \int_\Omega A_n\nabla\bar{u}_n\cdot\nabla(u_n-\bar{u}_n)^-\,dx - \langle f_n,(u_n-\bar{u}_n)^-\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} \\ &= \int_\Omega A_n\nabla\bar{u}_n\cdot\nabla(u_n-\bar{u}_n)^-\,dx + o(1) \\ &= \int_\Omega A^*\nabla u\cdot\nabla(u_n-\bar{u}_n)^-\,dx + o(1) = o(1). \end{aligned} \tag{4.17}$$

This combined with the equi-coerciveness of A_n implies that $\nabla(u_n-\bar{u}_n)^-$ strongly converges in $L^2(\Omega)^d$, which concludes the first step.

Second step: Proof of (4.7).
Set

$$\nu_n := \|u_n-\bar{u}_n\|_{H^1(\Omega)} \quad \text{and} \quad v_n := \frac{u_n-\bar{u}_n}{\tau_n+\nu_n}. \tag{4.18}$$

The sequence ν_n converges to 0 by the first step and v_n is bounded in $H^1_0(\Omega)$. Let us consider a positive sequence ε_n such that

$$\lim_{n\to+\infty}\varepsilon_n = \lim_{n\to+\infty}\frac{\nu_n}{\varepsilon_n} = \lim_{n\to+\infty}\frac{\tau_n\,k_n^2}{\varepsilon_n} = 0. \tag{4.19}$$

Such a sequence ε_n exists since ν_n and $\tau_n\,k_n^2$ converge to 0.

Now, let us study the set $\{u_n-(1-\varepsilon_n)\,u<0\}$. Since $(t\mapsto t^-)$ is 1-Lipschitz, we have by the definition (4.8) of $\bar{u}_n$

$$(u_n-u)^- \le (u_n-\bar{u}_n)^- + |\bar{u}_n-u| \le (u_n-\bar{u}_n)^- + d\,\tau_n\,k_n^2,$$

whence

$$\begin{aligned} u_n-(1-\varepsilon_n)\,u<0 \quad &\Longrightarrow \quad -(u_n-u)^- + \varepsilon_n\,u < 0 \\ &\Longrightarrow \quad -(u_n-\bar{u}_n)^- - d\,\tau_n\,k_n^2 + \varepsilon_n\,u < 0. \end{aligned}$$

This combined with the definition (4.18) of v_n yields

$$\{u_n-(1-\varepsilon_n)\,u<0\} \subset E_n := \left\{-\left(\frac{\tau_n+\nu_n}{\varepsilon_n}\right)v_n - \frac{d\,\tau_n\,k_n^2}{\varepsilon_n} + u < 0\right\}. \tag{4.20}$$

Finally, let us prove that $(u_n - (1-\varepsilon_n)\,u)^-$ strongly converges to 0 in $H^1_0(\Omega)$. On the one hand, proceeding as in (4.17) yields

$$
\begin{aligned}
&\alpha\,\|\nabla\,(u_n - (1-\varepsilon_n)\,u)^-\,\|^2_{L^2(\Omega)}\\
&\le -\int_\Omega A_n\nabla\,(u_n - (1-\varepsilon_n)\,u)\cdot\nabla\,(u_n - (1-\varepsilon_n)\,u)^-\\
&\le (\varepsilon_n - 1)\int_\Omega A_n\nabla u\cdot\nabla\,(u_n - (1-\varepsilon_n)\,u)\,\mathbf{1}_{\{u_n-(1-\varepsilon_n)\,u<0\}} + o(1)\\
&\le c\,\|\nabla u\,\mathbf{1}_{\{u_n-(1-\varepsilon_n)\,u<0\}}\|_{L^2(\Omega)},
\end{aligned}
$$

whence by taking into account the inclusion (4.20),

$$
\alpha\,\|\nabla\,(u_n - (1-\varepsilon_n)\,u)^-\,\|^2_{L^2(\Omega)} \le c\,\|\nabla u\,\mathbf{1}_{E_n}\|_{L^2(\Omega)}. \tag{4.21}
$$

On the other hand, since $u \ge 0$ a.e in Ω, we have $\nabla u\,\mathbf{1}_{E_n} = \nabla u\,\mathbf{1}_{E_n\cap\{u>0\}}$ a.e. in Ω. Moreover, in the definition (4.20) of E_n the sequence v_n converges a.e. in Ω (up to a subsequence) to some function in $H^1_0(\Omega)$. Then, thanks to the limits (4.19) satisfied by ε_n, the sequence $\mathbf{1}_{E_n\cap\{u>0\}}$ converges to 0 a.e. in Ω. Therefore, by the Lebesgue dominated convergence theorem the sequence $\nabla u\,\mathbf{1}_{E_n}$ strongly converges to 0 in $L^2(\Omega)^d$. This combined with estimate (4.21) yields the strong convergence (4.7).

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